Get an implied correlation to price equity-interest rates hybrids

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1 Introduction

1. A "hybrid" is a derivative on several underlyings which do not belong to the same asset class. The number of hybrid structures are about to rise drastically. Indeed such kind of pay-outs are easy to market since one can offer the performance of indexes -or equities- underlyings combined with some reassuring interest rates features. Nevertheless those products are not easily tractable. Defining the accurate amount needed to hedge the correlation risk between the different asset class underlyings brings about thinking "hybrid".

- As of today most of the hybrids products are priced using historical correlations as imputs. Technical margins are taken to insure the trading not to loose money on this "non-yet-hedgeable" parameter. Knowing an implied correlation, if not a way to hedge this risk, would be a manner to size the "risk of model" margin. Once the correlation calibrated, one can handle more accurately the pricing of products depending on the interest rates and the volatility of whatever equity underlyings. However, if one considers indexes as macro-economic variables, a causality relation is likely to exist between interest rates and indexes. Therefore, we will focus on products linked to interest-rates and to indexes. The correlation between those assets would be an intuitive parameter.
- Usually, one gets implied parameters by extracting the prices of the instruments one will buy to hedge its position from the market. However no straight-forward technics -contracts- are at use to hedge the hybrid correlation risk. Taking this limitation into account, we suggest in this article to get an implied correlation assuming that consistency exists between prices of different asset class' financial instruments.

2. Ingersoll (19) was one of the first to propose a paper on cross-asset underlyings option pricing. Carayannopoulos (20) did an empirical study on the valuation of convertible bonds under stochastic interest rates. Geman- El Karoui- Rochet introduced the theoretical materials allowing the cross-asset thinking in (5). Since then, Brigo-Mercurio has devoted an appendice in (1) to hybrids and Overhaus has written a survey on Equity hybrid derivatives (6). Recently, a technological break-through was made by the Ito33's research team on the pricing of convertible bonds. Ayache, Forsyth and Vetzal (22) explained what was realised on this area. Das and Sundaram (16) proposed a simple model taking into account the equity, the interest rates and the default risks.

3. In this article, we present a simple way to extract implied correlation from different markets using an assumed consistency between prices observed on different asset class markets.

   - We assume the volatility of the interest rates to be the same when pricing caps/floors and long dated index options. We assume as well the expectation of the variance to be the same when pricing variance swaps and long dated index options. We suggest to interpolate the information between different markets using a stochastic interest-rates stochastic volatility model. The underlying dynamic will be expressed in the T-forward risk neutral measure to allow the calibration of the implied correlation to market data.
   - We use a Hull-White dynamic to model the stochastic interest rates and a Cox-Ingersoll-Ross dynamic for the volatility of the index. We calibrate on each relevant market a first set of parameters. The caps market for the interest rates linked parameters and the variance swap market to extract the index variance linked ones. Once this first set of parameters obtained, we calibrate the vol of vol, the correlation between the underlying and the interest rates risk factor, the correlation between the underlying and the volatility risk factor and finally the correlation between the volatility and the interest rates risk factor.
   - Considering the numerical resolution, we made an other assumption. In our discretization scheme, when we solve the stochastic differential equation, we assume the stochastic volatility to be piecewise constant. This approximation gives good results if, and only if, the discretization time step is small enough. We calibrate our model using monte carlo simulations. We use antithetic variables technics as much as low dicrepencies Sobol sequences. We calibrate our model using the adaptative simulated annealing algorithm developped by Ingdberg (17) and (23).

4. This article presents a simple way to extract implied correlation in order to price hybrid structured products.
   - The model obtained after the calibration process is able to re-price accurately vanilla instruments of differents asset class. Therefore we are able to frame the hybrid correlation risk.
   - We could also infer that vol of vol might be structurally over-estimated. When calibrating a Heston model with Carr-Madan technics on the same set of market data as the one we use to calibrate the hybrid model, we find that vol of vol is significantly higher. The stochastic interest rates might explain a part of the smile. Consequently vol of vol would not need to be as hight. Nevertheless we do not show it in this paper.
5. This paper will be organized in three parts. First, the presentation of the model expressed in the T-forward risk neutral measure used in the calibration algorithm. Then, we will present our numerical results. As a conclusion, we will compare the results given by the calibration of a Heston model on the same set of data and the results we previously found.

2 The stochastic interest rates - stochastic volatility model

We will first solve the stochastic differential equation (SDE). Then we will express our results in the T-forward risk neutral measure and finally we will present our discretization scheme used in the calibration algorithm.

2.1 The dynamics

We assume the following dynamics for interest rates, the underlying and the volatility of the underlying respectively. W_t^i is a correlated multudimensional brownian motion (BM).

$$\begin{cases} dr_t = (\nu(t) - \lambda r_t)dt + vol.dW_t^a \\ \frac{dS_t}{S_t} = (r_t - d)dt + \sqrt{V_t}dW_t^b \\ dV_t = \kappa(\theta - V_t)dt + \sigma\sqrt{V_t}dW_t^c \end{cases} \tag{1}$$

with

$$\begin{cases} < dW_t^a, dW_t^b >= cor.dt \\ < dW_t^a, dW_t^c >= corvol.dt \\ < dW_t^b, dW_t^c >= \rho dt \end{cases} \tag{2}$$

We model the interest rates in the framework proposed by Hull-White (and Brigo-Mercurio). The variance V_t follows the familiar square-root process [used by Cox, Ingersoll, and Ross (CIR) (1985)]. The underlying's dynamic looks like the one presented in the Heston's paper of 1993 with a stochastic interast rate (following a Hull-White short rate dynamic). To illustrate our methodology :

$$dS_t = S_t r_t dt + S_t \sigma_t dW_t$$

Ito's lemma shows that:

$$dln(S_t) = (r_t - \frac{\sigma_t^2}{2})dt + \sigma_t dW_t$$

So that:

$$S_T = S_t exp\left\{\int_t^T (r_u - \frac{\sigma_u^2}{2})du + \int_t^T \sigma_u dW_u\right\}$$

One does not know how to express σ_t when express as a CIR process. To cope with it that we will assume σ_t to be piecewise constant. Thus we can use a Milstein discretization scheme for the varaince process V_t. Therefore, with $\Delta_{t_i} = t_{i+1} - t_i$, the underlying is expressed as

$$S_{t_{i+1}} = \begin{cases} S_{t_i} exp\{\int_{t_i}^{t_{i+1}} (r_u - d - \frac{V_u}{2})du + \int_{t_i}^{t_{i+1}} \sqrt{V_u} dW_u\} \\ \\ V_{t_{i+1}} = V_{t_i} + \kappa(\theta - V_{t_i})\Delta t + \sigma\sqrt{V_{t_i}} W_{\Delta t_i} + \frac{\sigma^2}{4}(W_{\Delta t_i}^2 - \Delta t_i) \end{cases} \tag{3}$$

We then obtain

$$S_{t_{i+1}} = \begin{cases} S_{t_i} exp\{\int_{t_i}^{t_{i+1}} r_u du - (d + \frac{V_{t_i}}{2})\Delta t_i + \sqrt{V_{t_i}} W_{\Delta t_i}\} \\ \\ V_{t_{i+1}} = V_{t_i} + \kappa(\theta - V_{t_i})\Delta t_i + \sigma\sqrt{V_{t_i}} W_{\Delta t_i} + \frac{\sigma^2}{4}(W_{\Delta t_i}^2 - \Delta t_i) \end{cases} \tag{4}$$

2.2 The dynamic under the T-forward risk neutral measure

To be able to calibrate on market data, one needs to move to a forward risk neutral measure, Q^T. Indeed, the market does not give you any hint on the way r_t, the short-instantaneous-rate, will behave between t_i andt_{i+1}. Thus one discounts the future cash-flows using zero-coupon bonds (ZCB) observed on the market. However, doing it, one has to express their dynamics under the probabilistic measure relative to the numeraire ZCB chosen.

1. The change of measure One can find a similar development in Brigo-Mercurio. We choose the ZCB maturing at T B(t,T) as a numeraire.

$$\frac{dQ^T}{dQ} = \frac{exp\{-\int_0^T r_u du\}}{B(0,T)} \tag{5}$$

We know that we can rewrite r_t as

$$r_t = x_t + \phi(t)$$

with

$$\phi(t) = f^M(0,T) + \frac{\sigma^2}{2\lambda}(1 - e^{\lambda T})^2$$

and

$$f^M(0,T) = -\frac{\partial ln(B^M(0,T))}{\partial T}$$

and that the term structure of discount factors, the ZCB, currently observed in the market is given by the sufficiently smooth function $T \rightarrow B^M(0,T)$. After solving a simple SDE, we find

$$x_t = e^{-\lambda t} vol \int_0^t e^{\lambda u} dW_u$$

The Radon-Nikodym derivatives becomes (see appendix 1 for the details)

$$\frac{dQ^T}{dQ} = \frac{exp\{-\int_0^T x_u du - \int_0^T \phi(u) du\}}{B(0,T)} \tag{6}$$

then,

$$\frac{dQ^T}{dQ} = \frac{exp\left\{\frac{vol}{\lambda}\int_0^T (1 - e^{\lambda(T-u)}) dW_u - \int_0^T f^M(0,u) du - \int_0^T \frac{vol^2}{2\lambda^2}(1 - e^{\lambda(T-u)})^2 du\right\}}{B(0,T)} \tag{7}$$

and finally,

$$\frac{dQ^T}{dQ} = exp\left\{\frac{vol}{\lambda}\int_0^T (1 - e^{\lambda(T-u)}) dW_u - \int_0^T \frac{vol^2}{2\lambda^2}(1 - e^{\lambda(T-u)})^2 du\right\} \tag{8}$$

The Girsanov theorem implies that

$$dW_t^T = dW_t + \frac{vol}{\lambda}\int_0^T (1 - e^{\lambda(T-u)}) dt$$

Note that we have only been working on the brownian motion W_t^a. The two other BM are not affected by this change of measure. To be able to facilitate the computation, one would better work with independent BM.

2. The Cholesky decomposition
$\bar{W}_t^i$ are independent BM and we have the following correlation matrix

$$C = \overbrace{\begin{pmatrix} 1 & cor & corvol \\ cor & 1 & \rho \\ corvol & \rho & 1 \end{pmatrix}}^{r_t \; S_t \; V_t}$$

We know that, by the Cholesky decomposition, $C = LL^T$ and that

$$L = \begin{pmatrix} 1 & 0 & 0 \\ cor & \sqrt{1-cor^2} & 0 \\ corvol & \frac{\rho - cor.corvol}{\sqrt{1-cor^2}} & \sqrt{1 - corvol^2 - \frac{(\rho - cor.corvol)^2}{1-cor^2}} \end{pmatrix}$$

Thus

$$\left\{ \begin{array}{c} dW_t^a = d\bar{W}_t^a \\ \\ dW_t^b = cor.d\bar{W}_t^a + \sqrt{1-cor^2}d\bar{W}_t^b \\ \\ dW_t^c = corvol.d\bar{W}_t^a + \frac{\rho - cor.corvol}{\sqrt{1-cor^2}}d\bar{W}_t^b + \sqrt{1 - corvol^2 - \frac{(\rho - cor.corvol)^2}{1-cor^2}}d\bar{W}_t^c \end{array} \right. \tag{9}$$

The variance's -V_t- dynamics under the risk neutral measure Q is

$$dV_t = \kappa(\theta - V_t)dt + \sigma\sqrt{V_t}dW_t^c$$

It becomes, again under Q, but with independent BM

$$dV_t = \kappa(\theta - V_t)dt + \sigma\sqrt{V_t}\left(corvol.d\bar{W}_t^a + \frac{\rho - cor.corvol}{\sqrt{1-cor^2}}d\bar{W}_t^b + \sqrt{1 - corvol^2 - \frac{(\rho - cor.corvol)^2}{1-cor^2}}d\bar{W}_t^c\right)$$

Hence, Girsanov gives, under Q^T

$$\begin{array}{c} dV_t = \left(\kappa(\theta - V_t) + \sigma\sqrt{V_t}corvol.\frac{vol}{\lambda}(1 - e^{-\lambda(T-t)}\right)dt \cdots \\ +\sigma\sqrt{V_t}\left(corvol.d\bar{W}_t^{Ta} + \frac{\rho - cor.corvol}{\sqrt{1-cor^2}}d\bar{W}_t^{Tb} + \sqrt{1 - corvol^2 - \frac{(\rho - cor.corvol)^2}{1-cor^2}}d\bar{W}_t^{Tc}\right) \end{array} \tag{10}$$

Hence, the T-forward risk neutral dynamic, expressed with independent BM -$\bar{W}_t^i$ -, is

$$\left\{\begin{array}{c} S_{t_{i+1}} = S_{t_i} exp\{\int_{t_i}^{t_{i+1}} r_u du - (d + \frac{V_{t_i}}{2})\Delta t_i - \sqrt{V_{t_i}} cor \frac{vol}{\lambda} \int_{t_i}^{t_{i+1}} (1 - e^{-\lambda T - u}) du \cdots \\ + \sqrt{V_{t_i}} cor.\bar{W}^{Ta}_{\Delta t_i} + \sqrt{V_{t_i}} \sqrt{1 - cor^2} \bar{W}^{Tb}_{\Delta t_i}\} \\ \\ V_{t_{i+1}} = V_{t_i} + [\sigma \sqrt{V_{t_i}} corvol. \frac{vol}{\lambda} (1 - e^{-\lambda(T - t_i)})] \Delta t_i \cdots \\ + \sigma \sqrt{V_{t_i}} corvol \bar{W}^{Ta}_{\Delta t_i} \cdots \\ + \sigma \sqrt{V_{t_i}} \frac{\rho - cor.corvol}{\sqrt{1 - cor^2}} \bar{W}^{Tb}_{\Delta t_i} \cdots \\ + \sigma \sqrt{V_{t_i}} \sqrt{1 - corvol^2 - \frac{(\rho - cor.corvol)^2}{1 - cor^2}} \bar{W}^{Tc}_{\Delta t_i} \cdots \\ + \frac{\sigma^2}{2} corvol \frac{\rho - cor.corvol}{\sqrt{1 - cor^2}} \bar{W}^{Ta}_{\Delta t_i} \bar{W}^{Tb}_{\Delta t_i} \cdots \\ + \frac{\sigma^2}{2} corvol \sqrt{1 - corvol^2 - \frac{(\rho - cor.corvol)^2}{1 - cor^2}} \bar{W}^{Ta}_{\Delta t_i} \bar{W}^{Tc}_{\Delta t_i} \cdots \\ + \frac{\sigma^2}{2} \frac{\rho - cor.corvol}{\sqrt{1 - cor^2}} \sqrt{1 - corvol^2 - \frac{(\rho - cor.corvol)^2}{1 - cor^2}} \bar{W}^{Tb}_{\Delta t_i} \bar{W}^{Tc}_{\Delta t_i} \cdots \\ - \frac{\sigma^2}{2} \Delta t_i [corvol^2 + (\frac{(\rho - cor.corvol)^2}{1 - cor^2})^2 + (1 - corvol^2 - \frac{(\rho - cor.corvol)^2}{1 - cor^2})] \end{array}\right. \tag{11}$$

Now, one has to calculate $\int_{t_i}^{t_{i+1}} r_u du$ under Q^T. One can refer to Brigo-Mercurio for a similar calculation.
We know that

$$r_t = x_p e^{-\lambda(t-p)} - \frac{vol^2}{\lambda} \int_p^t e^{-\lambda(t-u)} \left(1 - e^{-\lambda(T-u)}\right) du + vol \int_p^t e^{-\lambda(t-u)} d\bar{W}^{Ta}_t + \phi(t)$$

Hence

$$\int_{t_i}^{t_{i+1}} r_u du = \int_{t_i}^{t_{i+1}} \left[x_{t_i} e^{-\lambda(u-t_i)} - \frac{vol^2}{\lambda} \int_{t_i}^u e^{-\lambda(u-s)} \left(1 - e^{-\lambda(T-s)}\right) ds + vol \int_{t_i}^u e^{-\lambda(u-v)} d\bar{W}^{Ta}_v + \phi(u)\right] du$$

After some calculations (see appendix 2), we obtain

$$\begin{gathered}\int_{t_i}^{t_{i+1}} r_u du = \\ \frac{x_{t_i}}{\lambda}\left(1-e^{-\lambda\Delta_{t_i}}\right)+\frac{vol}{\lambda}\int_{t_i}^{t_{i+1}}\left(1-e^{-\lambda(t_{i+1}-u)}\right)d\bar{W}_u^{Ta}\cdots \\ -\frac{vol^2}{\lambda}\int_{t_i}^{t_{i+1}}\left[\int_{t_i}^{u}e^{-\lambda(u-s)}\left(1-e^{-\lambda(T-s)}\right)ds\right]du+\int_{t_i}^{t_{i+1}}f^M(0,u)du\cdots \\ +\frac{vol^2}{2\lambda^2}\int_{t_i}^{t_{i+1}}\left(1-e^{-\lambda u}\right)^2du\end{gathered} \tag{12}$$

3 Calculations

1. Appendix
   By an integration by parts, we have under Q

$$\int_t^T x_u du = (T-t)x_t + \underbrace{\int_t^T (T-u)dx_u}_{*}$$

Then

$$* = \int_t^T (T-u)dx_u = \int_t^T (T-u)(-\lambda x_u du + vol.dW_u^a) = \underbrace{-\lambda\int_t^T (T-u)x_u du}_{*\mathrm{a}} + \underbrace{vol\int_t^T (T-u)dW_u^a}_{*\mathrm{b}}$$

one needs to calculate

$$*\mathrm{a} = -\lambda\int_t^T (T-u)x_u du = \underbrace{-x_t\int_t^T \lambda(T-u)e^{-\lambda(u-t)}du}_{*\mathrm{a1}} + \underbrace{\lambda vol\int_t^T (T-u)\left(\int_t^u e^{-\lambda(u-s)}dW_u^a\right)du}_{*\mathrm{a2}}$$

We find

$$*\mathrm{a1} = -x_t\int_t^T \lambda(T-u)e^{-\lambda(u-t)}du = -x_t(T-t) - \frac{e^{-\lambda(T-t)}-1}{\lambda}x_t$$

and last but not least

$$\begin{gathered}*\mathrm{a2} = \lambda vol\int_t^T (T-u)\left(\int_t^u e^{-\lambda(u-s)}dW_u^a\right)du \\ = -\lambda vol\int_t^T e^{\lambda u}\left(\int_u^T (T-v)e^{-\lambda v}dv\right)dW_u^a \\ = -vol\int_t^T \left[(T-u)+\frac{1}{\lambda}\left(e^{-\lambda(T-u)-1}\right)\right]dW_u^a\end{gathered}$$

2. Appendix

$$\int_{t_i}^{t_{i+1}} r_u du = \underbrace{\int_{t_i}^{t_{i+1}} x_u du}_{(1)} + \underbrace{\int_{t_i}^{t_{i+1}} \phi(u) du}_{(2)}$$

We need to calculate (1) and (2)

$$(2) = \int_{t_i}^{t_{i+1}} \phi(u) du = \int_{t_i}^{t_{i+1}} f^M(0,u) du + \frac{vol^2}{2\lambda^2} \int_{t_i}^{t_{i+1}} \left(1 - e^{-\lambda u}\right)^2$$

and, by an integration by parts, we have under Q^T

$$(1) = \int_{t_i}^{t_{i+1}} x_u du = \Delta_{t_i} x_{t_i} + \underbrace{\int_{t_i}^{t_{i+1}} (t_{i+1} - u) dx_u}_{(1)^*}$$

Then

$$(1)* = \int_{t_i}^{t_{i+1}} (t_{i+1} - u) dx_u = \underbrace{\int_{t_i}^{t_{i+1}} (t_{i+1} - u) \left(-\lambda x_u du + vol.d\bar{W}_u^{Ta}\right)}_{(1)^*\text{a}} \underbrace{- \frac{vol^2}{\lambda} \int_{t_i}^{t_{i+1}} (t_{i+1} - u) \left(1 - e^{-\lambda(T-u)}\right) du}_{(1)^*\text{b}}$$

one needs now to calculate

$$(1)^*\text{a} = \textstyle\int_{t_i}^{t_{i+1}} (t_{i+1} - u) \left(-\lambda x_u du + vol.d\bar{W}_u^{Ta}\right)$$

$$= \underbrace{x_{t_i} \int_{t_i}^{t_{i+1}} -\lambda (t_{i+1} - u) e^{-\lambda(u - t_i)} du}_{(1)^*\text{a I}}$$

$$\underbrace{-\lambda vol \int_{t_i}^{t_{i+1}} \left[t_{i+1} - u \int_{t_i}^{u} e^{-\lambda(u-v)} d\bar{W}_v^{Ta} \right] du}_{(1)^*\text{a II}}$$

$$\underbrace{+vol^2 \int_{t_i}^{t_{i+1}} \left[t_{i+1} - u \int_{t_i}^{u} e^{-\lambda(u-v)} \left(1 - e^{-\lambda(T-v)}\right) dv \right] du}_{(1)^*\text{a III}}$$

Besides

$$(1)^*\text{b} = -\tfrac{vol^2}{\lambda}\int_{t_i}^{t_{i+1}}(t_{i+1}-u)\left(1-e^{-\lambda(T-u)}\right)du$$

$$= \underbrace{-\frac{vol^2}{\lambda}\int_{t_i}^{t_{i+1}}(t_{i+1}-u)du}_{(1)^*\text{b I}} + \underbrace{\frac{vol^2}{\lambda}\int_{t_i}^{t_{i+1}}(t_{i+1}-u)e^{-\lambda(T-u)}du}_{(1)^*\text{b II}}$$

4 Calibration

To calibrate the model one has to follow three steps. First the calibration of κ, V_0 and θ on the var swaps prices. Then, the calibration of λ and vol on the caps/floors prices. And finally we extract cor, corvol, σ and ρ from "vanilla" options prices. The model is calibrated on the volatility surface of the DAX based on the 6th september 2007 closing prices of vanilla options, the the 6th september 2007 closing prices of Caps/floors options on euribor 12 months and the 6th september 2007 variance swaps on DAX closing prices. The Dax index is a total return index.

4.1 Variance Swaps

Let's recall that in the stochastic interest rates - stochastic volatility model, instantaneous variance V_t follows the process

$$dV_t = \left(\kappa(\theta - V_t) + \sigma\sqrt{V_t}corvol.\tfrac{vol}{\lambda}(1-e^{-\lambda(T-t)})\right)dt\cdots$$

$$+\sigma\sqrt{V_t}\left(corvol.d\bar{W}_t^{Ta} + \tfrac{\rho - cor.corvol}{\sqrt{1-cor^2}}d\bar{W}_t^{Tb} + \sqrt{1-corvol^2-\tfrac{(\rho-cor.corvol)^2}{1-cor^2}}d\bar{W}_t^{Tc}\right)$$

Let's rewrite

$$\begin{cases} \alpha_1 dW_t^1 = \sigma\sqrt{V_t}\left(corvol.d\bar{W}_t^{Ta}\right) \\ \alpha_2 dW_t^2 = \tfrac{\rho-cor.corvol}{\sqrt{1-cor^2}}d\bar{W}_t^{Tb} \\ \alpha_3 dW_t^3 = \sqrt{1-corvol^2-\tfrac{(\rho-cor.corvol)^2}{1-cor^2}}d\bar{W}_t^{Tc} \end{cases}$$

We want this model to give the "fair variance strike" one can observe on the market. We need to be able to calculate the expectation of the total variance TV_T

$$E[W_T] = E\left[\int_0^T V_t dt\right]$$

Ito gives

$$d\left(e^{\kappa t}\right) = \kappa e^{\kappa t}V_t dt + e^{\kappa t}dV_t$$

$$= \kappa\theta e^{\kappa t}dt + e^{\kappa t}\sigma\sqrt{V_t}\left(adW_a + bdW_b + cdW_c\right)$$

hence

$$V_t = V_0 e^{-\kappa t} + \theta\left(1 - e^{-\kappa t}\right) + e^{-\kappa t}\sum_{i=1}^{3}\int_0^t e^{\kappa} s\sigma\sqrt{V_s}\alpha_i dW_t^i$$

Under the following conditions

$$E\left[\left(e^{-\kappa t}\int_0^t e^{\kappa s}\sqrt{V_s}\alpha_i dW_s^i\right)^2\right] < +\infty$$

We have

$$E[V_t] = V_0 e^{-\kappa t} + \theta\left(1 - e^{-\kappa t}\right)$$

and as a consequence, by Fubini

$$E[W_t] = E[\textstyle\int_0^t V_t] = \int_0^T E[V_t]dt$$

$$= \textstyle\int_0^t \left[V_0 e^{-\kappa t} + \theta\left(1 - e^{-\kappa t}\right)\right] dt$$

$$= \tfrac{\left(1-e^{-\kappa t}\right)}{\kappa}(V_0 - \theta) + \theta T$$

Finally, the expected annualized variance is given by

$$\frac{1}{T}E[W_T] = \frac{\left(1 - e^{-\kappa t}\right)}{\kappa T}(V_0 - \theta) + \theta$$

One notes that the expected annualized variance does not change with σ. Indeed a variance swap is replicated unically with vanilla options. As a consequence it is not "path dependent". We assume θ, the long term variance, to be the fair strike of the 5Y variance swap. After calibrating, we find

κ	θ	**v0**
9.0001	0.0687	0.0248

Kasyan (2007) shown that the functions of variance swap's prices were montonous. In the market configuration we are looking at, the term structure of the fair strike of the variance swaps, is not monotonous. As the consequence, the "best fit" we obtain does not match the entire curve.

Maturities	**Market**	**Model**
3M	23.84	22.64
6M	23.05	24.3
12M	23	25.27
5Y	26.22	26.03

4.2 Caps/Floors

One has to calibrate a Hull-White (1994) one factor model on Caps/Floors on Euribor 1M. Let's recall that

$$dr_t = (\nu(t) - \lambda r_t)dt + vol.dW_t^a$$

with
$\Phi(x)$ being the normal cumulative function and according to Brigo-Mercurio, a cap price is given by:

"We denote by D=$\{d_1, d_2, ..., d_n\}$ the set of the cap/floor payment dates and by $T = \{t_0, t_1, ..., t_n\}$ the set of the corresponding times, meaning that t_i is the difference in years between d_i and the settlement date t, and where t_0 is the first reset time. Moreover, we denote by τ_i the year fraction from d_{i-1} to d_i, $i = 1, ..., n$. (...) We then obtain that the price at time $t < t_0$ of the cap with cap rate (strike) K, nominal value N and set of times T is given by"

$$cap(t, T, N, K) = N \sum_{i=1}^{n} \big(B(t, t_{i-1} N\Phi(-h_i + \sigma_p^i) - (1 + K\tau_i)B(t, t_i)\Phi(-h_i)) \big)$$

with

$$\sigma_p^i = vol\sqrt{\frac{1 - e^{-2\lambda(t_{i-1} - t_i)}}{2\lambda}} B(t_{i-1}, t_i)$$

and

$$h_i = \frac{1}{\sigma_p^i} ln \left(\frac{B(t, t_i)(1 + K\tau_i}{B(t, t_{i-1})} + \frac{\sigma_p^i}{2} \right)$$

After calibrating, we find

λ	**vol**
0.0499	0.1972

and

Maturities	Volatilities
1Y	15.039
18M	15.572
2Y	15.6245
3Y	15.5361
4Y	15.3345
5Y	15.1328
6Y	14.8216
7Y	14.5278
8Y	14.2291
9Y	14.0322
10Y	13.7199
12Y	13.1719
15Y	12.7407
20Y	12.1378
25Y	11.6651
30Y	11.3864

finally

Maturities	Market Black	Model HW
1Y	0.001	0.0007
18M	0.0026	0.0022
2Y	0.0046	0.004
3Y	0.0084	0.0074
4Y	0.012	0.011
5Y	0.0158	0.0147
6Y	0.0193	0.0183
7Y	0.0227	0.0221
8Y	0.026	0.0258
9Y	0.0296	0.0296
10Y	0.0334	0.0336
12Y	0.0412	0.0416
15Y	0.0522	0.0535
20Y	0.0696	0.0711
25Y	0.0852	0.0855
30Y	0.0988	0.0973

4.3 Implied Correlation

We assume $corvol = 0$, and calibrate the model on the volatility surface of the Dax as of closing 06-Sept-2007

Maturities	5500	6500	7450	8400	9600	11000
90 d	36.27	29.85	23.56	17.41	15.69	17.23
180 d	32.07	27.45	22.88	18.16	15.98	16.6
360 d	29.33	26.17	22.97	19.55	17.07	16.05
565 d	28.26	25.86	23.43	20.80	18.47	17.25
1020 d	27.96	25.79	23.61	21.15	18.93	17.61

and we find

σ	ρ	**corr**
0.4033	-0.7464	0.0413

5 Impacts on vol of vol

One can calibrate Heston's stochastic volatility model on vanilla prices using Carr-Madan technics and get κ, θ, ρ and σ. Carr-Madan (1999) shows that, quoting Moodley (2005)
"$x_t \equiv lnS_t$ and $k \equiv lnK$, where K is the strike price of the option. The value of an european call, with maturity T becomes"

$$C_T(k) = e^{-rT} \int_k^{\infty} \left(e^{x_T} - e^k\right) f_T(x_T) dx_T$$

and the solution enables us to use FFT

$$C_T(k_u) \approx \frac{e^{-\alpha k_u}}{\pi} \sum_{j=1}^{N} e^{-i\frac{2\pi}{N}(j-1)(u-1)} e^{ibv_j} F_{c_T}(v_j) \frac{\eta}{3}(3 + (-1)^j - \delta_{j-1})$$

where,

$$v_j = \eta(j-1);$$
$$\eta = \frac{c}{N};$$
$$c = 600, N = 4096;$$
$$b = \frac{\pi}{\eta};$$
$$k_u = -b + \frac{2b}{N}(u-1);$$
$$u = 1, 2, ..., N+1$$

We recall that $F_{C_T}(\phi)$ is the characteristic function of x_T, and that under Q, given by (Hong 2004)

$$F_{C_T}(\phi) = e^{A(\phi)+B(\phi)+C(\phi)}$$

$$A(\phi) = i\phi(x_0 + rT)$$

$$B(\phi) = \frac{2\zeta(\phi)(1 - e^{-\psi(\phi)T}V_0)}{2\psi(\phi) - (\psi(\phi) - \gamma(\phi))(1 - e^{-\psi(\phi)T})}$$

$$C(\phi) = -\tfrac{\kappa\theta}{\sigma^2}\left[2log\left(\tfrac{2\psi(\phi) - (\psi(\phi)-\gamma(\phi))(1-e^{-\psi(\phi)T})}{2\psi(\phi)}\right) + (\psi(\phi) - \gamma(\phi))T\right]$$

$$\zeta(\phi) = -\frac{1}{2}(\phi^2 + i\phi)$$

$$\psi(\phi) = \sqrt{\gamma(\phi)^2 - 2\sigma^2\zeta(\phi)}$$

$$\gamma(\phi) = \kappa - \rho\sigma\phi i$$

Carr-Madan defined a modified call price function, $c_T(k)$

$$c_T(k) = e^{\alpha k}C_T(k), \alpha > 0$$

So that, the characteristic function of this modified call function is $F_{c_T}(\phi)$

$$F_{c_T}(\phi) = \int_{-\infty}^{\infty} e^{i\phi k}c_T(k)dk$$

However, "$c_T(k)$ is expected to be square integrable for a range of α values and $\forall k$". α is called the dampening factor. We need to choose α since it is an input of the calibration algorithm. (Carr-Madan 1999) suggest that α be chosen such that

$$E\left[S_T^{\alpha+1}\right] < \infty \Rightarrow F_{C_T}(-(\alpha+1)i) < \infty$$

(Lord-Kahl 2006) shown that, when looking at an optimal contour of the Fourier integral, taking into account numerical issues such as cancellation and explosion of the characteristic function, the range $\S \equiv (\alpha_-, \alpha_+)$ which contains all allowed corresponds to $(\zeta_- - 1, \zeta_+ - 1)$ with the maximum allowed ζ_+ for highly negative correlation $\rho \approx -1$ can be approximated by

$$\zeta_\pm \approx \frac{\sigma - 2\kappa\rho \pm \sqrt{(\sigma - 2\kappa\rho)^2 + 4(1-\rho^2)(\kappa^2 + 4\pi^2\tau^{-2})}}{2\sigma(1-\rho^2)}$$

Having defined the allowed range of α, (Lord-Kahl 2006) specified the optimal rule to choose α

$$\alpha^* = \underbrace{argmin}_{\alpha \in \S}\left[-\alpha K + ln(F_{C_T}(-(\alpha+1)i))\right]$$

one can finally calibrate the Heston model on the same market data and find

κ	θ	ρ	σ	**v0**
8.84	0.0189	-0.927	0.663	0.0087

6 Results and Analysis

Using a personnal PC and a matlab code, you would be able to get a price with the hybrid model stochastic interest rates stochastic volatility for a vanilla option with a standart deviation less than 10^2 in 90 seconds.

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